

Some Properties of Mannheim Curves in Galilean and Pseudo - Galilean space**Alper Osman ÖĞRENMİŞ, Handan ÖZTEKİN, Mahmut ERGÜT**

Fırat University, Science Faculty, Mathematics Department

23119 Elazığ / TÜRKİYE

ogrenmisalper@gmail.com, handanoztekin@gmail.com, mergut@firat.edu.tr

Abstract

The aim of this work is to study the Mannheim curves in 3-dimensional Galilean and Pseudo - Galilean space. We obtain the characterizations between the curvatures and torsions of the Mannheim partner curves.

Keywords: Galilean space, Pseudo-Galilean Space, Frenet frame, Mannheim curve, Curvature, Torsion.

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1. Introduction.

In classical differential geometry, there are a lot of studies on Bertrand and Mannheim curves. Many interesting results on Bertrand and Mannheim curves have been obtained by many mathematicians (see [2,7,8,9,10,11,13]). We can see in most studies, a characteristic property of Bertrand and Mannheim curves which asserts the existence of a linear relation between curvature and torsions. There are a lot of works on Bertrand curves but there are rather a few works on Mannheim curves.

In this study, we have done a study about some special curves in Galilean and Pseudo - Galilean space. However, to the best of author's knowledge, Mannheim curves has not been presented Galilean and Pseudo - Galilean space in depth. Thus, the study is proposed to serve such a need.

Our study is organized as follows. In section 2, some fundamental properties of Galilean and Pseudo - Galilean space are given which will be used in the later sections. In section 3, we give some characterizations of Mannheim curves in Galilean space. We also give some properties of Mannheim curves in Pseudo - Galilean space in section 4.

2. Basic notions and properties

The Galilean space is a three dimensional complex projective space P_3 in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points $I_1, I_2 \in f$ (the absolute points).

We shall take, as a real model of the space G_3 , a real projective space P_3 with the absolute $\{w, f\}$ consisting of a real plane $w \subset G_3$ and a real line $f \subset w$ on which an elliptic involution ε has been defined. We introduce homogeneous coordinates in G_3 in such a way that the absolute plane w is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and elliptic involution by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the distance between the points $P_i = (x_i, y_i, z_i)$, $i = 1, 2$, is defined by

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2 \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}, & \text{if } x_1 = x_2 \end{cases}.$$

Let α be a curve in G_3 , defined by arclength $\alpha : I \rightarrow G_3$ and parametrized by the invariant parameter $s \in I$, given in the coordinate form

$$\alpha(s) = (s, y(s), z(s)).$$

Then the curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\kappa(s) = \sqrt{y''^2(s) + z''^2(s)}$$

$$\tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)}$$

and associated moving trihedron is given by

$$\begin{aligned} T(s) &= \alpha'(s) = (1, y'(s), z'(s)) \\ N(s) &= \frac{1}{\kappa(s)} \alpha''(s) = \frac{1}{\kappa(s)} (0, y''(s), z''(s)) \\ B(s) &= \frac{1}{\kappa(s)} (0, -z''(s), y''(s)). \end{aligned}$$

The vectors T, N, B are called the vectors of the tangent, principal normal and binormal line of α , respectively. For their derivatives the following Frenet formulas hold

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= \tau(s)B(s) \\ B'(s) &= -\tau(s)N(s). \end{aligned} \tag{2.1}$$

More about the Galilean geometry can be found in [12].

The Pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature $(0, 0, +, -)$, explained in [3]. The absolute of the Pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where w is the ideal (absolute) plane, f is line in w and I is the fixed hyperbolic involution of points of f .

As in [3], Pseudo-Galilean scalar product can be written as

$$\langle v_1, v_2 \rangle = \begin{cases} x_1 x_2 & \text{if } x_1 \neq 0 \vee x_2 \neq 0 \\ y_1 y_2 - z_1 z_2 & \text{if } x_1 = 0 \wedge x_2 = 0 \end{cases}$$

where $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$.

It leaves invariant the Pseudo-Galilean norm of the vector $v = (x, y, z)$ defined by

$$\|v\| = \begin{cases} x, & x \neq 0 \\ \sqrt{|y^2 - z^2|}, & x = 0. \end{cases}$$

A vector $v = (x, y, z)$ is said to be non-isotropic if $x \neq 0$. All unit non-isotropic vectors are the form $(1, y, z)$. For isotropic vectors $x = 0$ holds. There are four types of isotropic vectors: spacelike ($y^2 - z^2 > 0$), timelike ($y^2 - z^2 < 0$) and two types of lightlike ($y = \pm z$) vectors. A non-lightlike isotropic vector is a unit vector if $y^2 - z^2 = \pm 1$.

Let $\alpha : I \longrightarrow G_3^1$, $I \subset \mathbb{R}$ be a curve given by

$$\alpha(t) = (x(t), y(t), z(t)),$$

where $x(t), y(t), z(t) \in C^3$ (the set of three times continuously differentiable functions) and t run through a real interval [3].

A curve $\alpha(t)$ in G_3^1 is said to be admissible curve if $x'(t) \neq 0$ [3].

The curves in G_3^1 are characterized as follows [4,5]

Type I: Let α be an admissible curve in G_3^1 , parametrized by arclength $t = s$, given in coordinate form

$$\alpha(s) = (s, y(s), z(s)).$$

Then the curvature $\kappa_\alpha(s)$ and the torsion $\tau_\alpha(s)$ are defined by

$$\kappa_\alpha(s) = \sqrt{|y''^2(s) - z''^2(s)|}$$

$$\tau_\alpha(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa_\alpha^2(s)}$$

and associated moving trihedron is given by

$$\begin{aligned} T_\alpha(s) &= \alpha'(s) = (1, y'(s), z'(s)) \\ N_\alpha(s) &= \frac{1}{\kappa_\alpha(s)} \alpha''(s) = \frac{1}{\kappa_\alpha(s)} (0, y''(s), z''(s)) \\ B_\alpha(s) &= \frac{1}{\kappa_\alpha(s)} (0, z''(s), y''(s)). \end{aligned}$$

The vectors T_α , N_α , B_α are called the vectors of the tangent, principal normal and binormal line of α , respectively. For their derivatives the following Frenet formulas hold

$$\begin{aligned} T'_\alpha(s) &= \kappa_\alpha(s)N_\alpha(s) \\ N'_\alpha(s) &= \tau_\alpha(s)B_\alpha(s) \\ B'_\alpha(s) &= -\tau_\alpha(s)N_\alpha(s) \end{aligned} \tag{2.2}$$

Type II: Let β be an admissible curve in G_3^1 , parametrized by arclength s , given in coordinate form

$$\beta(s) = (s, y(s), 0).$$

Then the curvature $\kappa_\beta(s)$ and the torsion $\tau_\beta(s)$ are defined by

$$\begin{aligned} \kappa_\beta(s) &= y''(s) \\ \tau_\beta(s) &= \frac{a'_2(s)}{a_3(s)} \end{aligned}$$

and associated moving trihedron is given by

$$\begin{aligned} T_\beta(s) &= (1, y'(s), 0), \\ N_\beta(s) &= (0, a_2(s), a_3(s)), \\ B_\beta(s) &= (0, a_3(s), a_2(s)). \end{aligned}$$

where, $y, a_2, a_3 \in C^\infty$, $s \in I \subseteq \mathbb{R}$.

The vectors T_β , N_β , B_β are called the vectors of the tangent, principal normal and binormal line of β , respectively. For their derivatives the following Frenet's formulas hold

$$\begin{aligned} T'_\beta(s) &= \kappa_\beta(s)(\cosh \phi(s)N_\beta(s) - \sinh \phi(s)B_\beta(s)), \\ N'_\beta(s) &= \tau_\beta(s)B_\beta(s), \\ B'_\beta(s) &= \tau_\beta(s)N_\beta(s), \end{aligned} \tag{2.3}$$

where $\phi(s)$ is the angle between $a(s) = (0, a_2(s), a_3(s))$ and the plane $z = 0$.

3. Mannheim Curves in Galilean Space

Definition 3.1. Let α and α_1 be curves in 3-dimensional Galilean space G_3 . If there exists a corresponding relationship between the space curves α and α_1 such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of α_1 , then α is called a Mannheim curve and α_1 is called a Mannheim partner curve of α . The pair $\{\alpha, \alpha_1\}$ is said to be a Mannheim pair. [6]

Remark 3.2. Let α be a curve in Galilean space G_3 . Then α is Mannheim curve if and only if its curvature κ_α and torsion τ_α satisfy the relation $\kappa_\alpha = c\tau_\alpha^2$ for some constant c . [10]

Theorem 3.3. Let α be a Mannheim curve in Galilean space G_3 . Then α_1 is the Mannheim partner curve of α . Then the curvature κ_1 and the torsion τ_1 of α_1 satisfy the following equation

$$\tau_1' = \frac{\kappa_1}{\lambda}(\lambda^2\tau_1^2 + 1) \quad (3.1)$$

for some nonzero constant λ .

Proof. Suppose that $\alpha(s)$ is a Mannheim curve in Galilean space G_3 . Then we can write

$$\alpha(s_1) = \alpha_1(s_1) + \lambda(s_1)B_1(s_1) \quad (3.2)$$

for some function $\lambda(s_1)$. By taking derivative of (3.2) with respect to s_1 and using the Frenet equations in Galilean space G_3 , we get

$$T \frac{ds}{ds_1} = T_1 + \lambda' B_1 - \lambda \tau_1 N_1. \quad (3.3)$$

Since B_1 is coincident with N , we obtain

$$\lambda'(s_1) = 0$$

that means that λ is nonzero constant. Thus, we have

$$T \frac{ds}{ds_1} = T_1 - \lambda \tau_1 N_1. \quad (3.4)$$

On the other hand we have

$$T = T_1 \cos \theta + N_1 \sin \theta \quad (3.5)$$

where θ is the angle between T and T_1 at the corresponding points of α and α_1 . Differentiating of (3.5) with respect to s_1 , we get

$$\begin{aligned} \kappa N \frac{ds}{ds_1} &= -T_1 \theta' \sin \theta + \kappa_1 N_1 \cos \theta + N_1 \theta' \cos \theta + \tau_1 B_1 \sin \theta \\ \kappa N \frac{ds}{ds_1} &= -T_1 \theta' \sin \theta + N_1 (\kappa_1 + \theta') \cos \theta + \tau_1 B_1 \sin \theta. \end{aligned} \quad (3.6)$$

Since $\{\alpha, \alpha_1\}$ is a Mannheim pair, we obtain

$$\kappa_1 + \theta' = 0$$

and therefore we have

$$\theta' = -\kappa_1. \quad (3.7)$$

From

(3.4) and (3.5), we find that

$$\lambda\tau_1 = -\tan\theta. \quad (3.8)$$

By taking derivative of this last equation and applying (3.7), we obtain

$$\lambda\tau_1' = -\theta'(1 + \tan^2\theta). \quad (3.9)$$

If we consider (3.7) and (3.8) in (3.9), we get

$$\tau_1' = \frac{\kappa_1}{\lambda}(\lambda^2\tau_1^2 + 1).$$

Hence the proof is completed.

Proposition 3.4. Let α be a Mannheim curve in Galilean space G_3 and α_1 be the Mannheim partner curve of α . If α is a generalized helix, then α_1 is a planar curve.

Proof. Let T, N, B the tangent, principal normal and binormal vector field of the curve α , respectively. From the properties of generalized helix and the definition of Mannheim curves in Galilean space G_3 , we have

$$\kappa < N, P > = 0$$

and

$$\kappa < B_1, P > = 0$$

for a constant direction P in Galilean space G_3 . Then it is easy to obtain that $\tau_1 = 0$.

4. Mannheim Curves in Pseudo - Galilean Space

Definition 4.1. Let α and α_1 be an admissible curves defined in type I with nonzero $\kappa_\alpha, \tau_\alpha$, $s \in I$ in Pseudo - Galilean space and $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_{\alpha_1}, N_{\alpha_1}, B_{\alpha_1}\}$ be Frenet frame in Pseudo - Galilean space G_3^1 along α and α_1 , respectively. If there exists a corresponding relationship between the admissible curves α and α_1 such that, at the corresponding points of the admissible curves, principal normal lines N_α of α coincides with the binormal lines B_{α_1} of α_1 , then α is called an admissible Mannheim curves and α_1 is called an admissible Mannheim partner curve of α . The pair $\{\alpha, \alpha_1\}$ is said to be an admissible Mannheim pair in Pseudo - Galilean space G_3^1 . [1]

Theorem 4.2. Let α an admissible curve defined in Type I in Pseudo - Galilean space G_3^1 . Then α is an admissible Mannheim curve if and only if its curvature κ_α and torsion τ_α satisfy the relation $\kappa_\alpha = -c\tau_\alpha^2$ for some constant c .

Proof. Let $\alpha = \alpha(s)$ be an admissible Mannheim curve. Let us denote by $\{T_\alpha, N_\alpha, B_\alpha\}$ the Frenet frame field of α .

Assume that $\alpha_1 = \alpha_1(s_1)$ is an admissible curve whose binormal direction coincides with the principal normal of α . Namely let us denote by $\{T_{\alpha_1}, N_{\alpha_1}, B_{\alpha_1}\}$ the Frenet frame field of α_1 . Then $B_{\alpha_1}(s_1) = \pm N_\alpha(s)$.

The curve α_1 is parametrized by arclength s as

$$\alpha_1(s) = \alpha(s) + \lambda(s)N_\alpha(s) \quad (4.1)$$

for some function $\lambda(s) \neq 0$. Differentiating (4.1) with respect to s , we find

$$\alpha_1' = T_\alpha + \lambda'N_\alpha + \lambda\tau_\alpha B_\alpha. \quad (4.2)$$

Since the binormal direction of α_1 coincides with the principal normal of α , we have $\lambda' = 0$. Hence λ is constant. The second derivative α_1'' with respect to s is

$$\alpha_1'' = (\kappa_\alpha + \lambda\tau_\alpha^2)N_\alpha + \lambda\tau_\alpha' B_\alpha. \quad (4.3)$$

Since N_α is in the binormal direction of α_1 , we have

$$\kappa_\alpha + \lambda\tau_\alpha^2 = 0.$$

Conversely, let α be an admissible curve. Then the curve

$$\alpha_1(s) = \alpha(s) + \lambda N_\alpha(s)$$

has binormal direction N_α .

Theorem 4.3. Let β an admissible curve defined in Type II in Pseudo - Galilean space G_3^1 . Then β is an admissible Mannheim curve if and only if its curvature κ_β and torsion τ_β satisfy the relation $\kappa_\beta = \frac{-c}{\cosh \phi} \tau_\beta^2$ for some constant c .

Proof. If we consider equations (2.3) and proof of the Theorem 4.2. we can prove the theorem easily.

Theorem 4.4. Let α be an admissible Mannheim curve defined by Type I in Pseudo - Galilean space G_3^1 . Then α_1 is the admissible Mannheim partner curve of α . Then the curvature κ_1 and the torsion τ_1 of α_1 satisfy the following equation

$$\tau_1' = \frac{\kappa_1}{\lambda} (\lambda^2 \tau_1^2 - 1) \quad (4.4)$$

for some nonzero constant λ .

Proof. It is similar to proof of theorem 3.3.

Remark 4.5. By a simple parameter transformation, the condition

$$\tau_1' = \frac{\kappa_1}{\lambda}(\lambda^2 \tau_1^2 - 1)$$

can be written as

$$\tau_1 = -\frac{\varepsilon}{\lambda} \tan(\varepsilon \int \kappa_1 ds + c_0).$$

Therefore, for each an admissible Mannheim curve in Pseudo - Galilean space G_3^1 , there is an unique Mannheim partner curve.

This reality is true for Mannheim curve in Galilean space G_3 .

Proposition 4.6.. Let α be an admissible Mannheim curve in Pseudo - Galilean space G_3^1 and α_1 be the admissible Mannheim partner curve of α . If α is a generalized helix, then α_1 is a planar curve.

Proof. Let T, N, B the tangent , principal normal and binormal vector field of the curve α , respectively. From the properties of generalized helix and the definition of admissible Mannheim curves in Pseudo - Galilean space G_3^1 , we have

$$\kappa < N, P > = 0$$

and

$$\kappa < B_1, P > = 0$$

for a constant direction P in Pseudo - Galilean space G_3^1 . Then it is easy to obtain that $\tau_1 = 0$.

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